

Existence and regularity properties of solutions to fractional nonlinear problems

The research project is aimed at candidates working on PDEs, free boundary problems, calculus of variations and geometric measure theory.

In particular, it will focus on existence and regularity results for nonlocal and nonlinear problems exhibiting growth conditions similar to the ones of the fractional p -Laplacian. This model arises as natural nonlocal counterpart to the classical p -Laplace equation considering the first variation of the fractional Gagliardo seminorm

$$[u]_{W^{s,p}(\mathbb{R}^d)} = \left(\int \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} dx dy \right)^{1/p}. \quad (1)$$

Minimizers of this seminorm under suitable conditions to be imposed in $\mathbb{R}^d \setminus \Omega$ (Dirichlet or Neumann boundary conditions) satisfy the fractional p -Laplace equation

$$(-\Delta_p)^s u(x) = \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^{p-2}}{|x - y|^{d+sp}} (u(x) - u(y)) dy = 0 \quad \text{for } x \in \Omega. \quad (2)$$

The aim of this project is twofold: on one side we are interested in studying existence and non-existence results of non-constant positive solutions for a nonlinear subcritical fractional equation with nonlocal Neumann boundary conditions; on the other hand we aim to study interior regularity properties of solutions to (2).

Concerning the first part of the project, the prototype problem that we will consider is given by:

$$\begin{cases} (-\varepsilon \Delta_p)^s u + u = u^q & \text{on } \Omega, \\ \mathcal{N}_s u = 0 & \text{in } \mathbb{R}^d \setminus \Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \quad (3)$$

where

$$\mathcal{N}_s u(x) = \int_{\Omega} \frac{|u(x) - u(y)|^{p-2}}{|x - y|^{d+sp}} (u(x) - u(y)) dy$$

denotes the nonlocal Neumann condition.

Some existence and non existence results (depending on the smallness of the diffusion parameter ε) have been obtained in the case $p = 2$ and for $1 < q < \frac{n}{n-2s}$. We aim to extend such results (including a crucial uniform L^∞ - estimate) to all subcritical powers $q < \frac{n+2s}{n-2s}$ and to the nonlinear case $p \neq 2$ for subcritical q .

A second question of utmost importance is the one about regularity of solutions. While the regularity theory for the linear case $p = 2$ is well understood, the case of the fractional p -Laplacian

is significantly more challenging to treat and some questions are still open. In particular, $C^{1,\alpha}$ regularity has been obtained recently in the range $s \in (0, 1)$ and $2 \leq p \leq \frac{2}{1-s}$. Building upon this result and using newly introduced techniques, we aim to extend it to the singular case where $p < 2$ and, possibly, to more general nonlinear nonlocal operators. We emphasize that regularity is not just interesting in itself but plays a crucial role also in connection with the first part of the project, since it is a crucial ingredient for getting uniform a-priori bounds of solutions via a blow-up argument.

We intend to develop the project according to the following timeline:

- we will start by considering the semilinear problem (3) for $p = 2$ and try to get L^∞ -estimate for solutions in the whole range $1 < q < \frac{n+2s}{n-2s}$. As a consequence we would get non-existence of non-constant solution if the diffusion coefficient is large enough. This would extend the results already known for $1 < q < \frac{n}{n-2s}$.
- in a second step, we would try to extend the above results to the nonlinear case $p \neq 2$. A possible approach in getting uniform L^∞ -estimates is via a Moser iteration. This probably will not allow to cover the whole range of subcritical q . Another approach, based on a blow-up argument, relies on regularity of solutions. For this reason the study of interior regularity for solutions to fractional p -Laplace equation will be fundamental.
- the third step focuses on the regularity theory in the nonlinear setting. Starting from a recent result which establishes $C^{1,\alpha}$ regularity of solutions to (2) for any $s \in (0, 1)$ and $2 \leq p \leq \frac{2}{1-s}$, we will try to consider the singular case $1 < p < 2$.

All. 1 – Piano delle attività e criteri di valutazione

Piano delle attività

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In particular, it will focus on existence and regularity results for nonlocal and nonlinear problems exhibiting growth conditions similar to the ones of the fractional p -Laplacian. This model arises as natural nonlocal counterpart to the classical p -Laplace equation considering the first variation of a fractional Gagliardo seminorm.

The aim of this project is twofold: on one side we are interested in studying existence and nonexistence results of non-constant positive solutions for a nonlinear subcritical fractional equation with nonlocal Neumann boundary conditions; on the other hand we aim to study interior regularity properties of solutions to the fractional p -Laplace equation.

We intend to develop the project according to the following timeline:

- we will start by considering a semilinear Neumann problem for the spectral fractional Laplacian and try to get uniform estimates for solutions in the whole range of subcritical nonlinearities. As a consequence, we would get non-existence of non-constant solution if the diffusion coefficient is large enough. This would extend some results, already known in some particular cases, to any subcritical nonlinearity;
- in a second step, we would try to extend the above results to the fractional p -Laplacian. A possible approach in getting uniform estimates is via a Moser iteration. This probably will not allow to cover the whole range of subcritical nonlinearities. Another approach, based on a blow-up argument, relies on regularity of solutions. For this reason the study of interior regularity for solutions to fractional p -Laplace equation will be fundamental.
- the third step focuses on the regularity theory in the nonlinear setting. Starting from a recent result which establishes $C^{1,\alpha}$ regularity of solutions to the fractional p -Laplace equation for $p > 2$, we will try to consider the singular case $1 < p < 2$.

• **Quantificazione figurativa delle attività annue, ai fini della rendicontazione dei costi sui progetti di ricerca:**

X 1720 ore annue

2) Criteri di valutazione, definiti sulla base di quanto stabilito all'art. 10 del Regolamento

a) coerenza del titolo di studio per l'accesso alla selezione con il progetto oggetto del bando	a) coerenza del titolo di studio per l'accesso alla selezione con il progetto oggetto del bando
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<i>(consistency of additional qualifications with the project covered by the call for applications)</i>	<i>(consistency of additional qualifications with the project covered by the call for applications)</i>
b) coerenza di ulteriori titoli di studio con il progetto oggetto del bando <i>(consistency of additional qualifications with the project covered by the call for applications)</i>	b) coerenza di ulteriori titoli di studio con il progetto oggetto del bando <i>(consistency of additional qualifications with the project covered by the call for applications)</i>
c) le pubblicazioni presentate in termini di originalità, innovatività, rigore metodologico, rilevanza e congruenza con il progetto oggetto del bando. La Commissione prende in considerazione esclusivamente pubblicazioni o testi accettati per la pubblicazione secondo le norme vigenti, nonché saggi inseriti in opere collettanee ed articoli editi su riviste in formato cartaceo o digitale con l'esclusione di note interne o rapporti dipartimentali <i>(the publications submitted in terms of originality, innovativeness, methodological rigour, relevance and congruence with the project covered by the call for applications. The Committee will only take into consideration publications or texts accepted for publication in accordance with the applicable regulations, as well as any essays included in collective works and articles published in journals in paper or digital format with the exclusion of internal notes or departmental reports)</i>	c) le pubblicazioni presentate in termini di originalità, innovatività, rigore metodologico, rilevanza e congruenza con il progetto oggetto del bando. La Commissione prende in considerazione esclusivamente pubblicazioni o testi accettati per la pubblicazione secondo le norme vigenti, nonché saggi inseriti in opere collettanee ed articoli editi su riviste in formato cartaceo o digitale con l'esclusione di note interne o rapporti dipartimentali <i>(the publications submitted in terms of originality, innovativeness, methodological rigour, relevance and congruence with the project covered by the call for applications. The Committee will only take into consideration publications or texts accepted for publication in accordance with the applicable regulations, as well as any essays included in collective works and articles published in journals in paper or digital format with the exclusion of internal notes or departmental reports)</i>
d) altri titoli collegati ad attività precedentemente svolte (es: borse di	d) altri titoli collegati ad attività precedentemente svolte (es: borse di

studio, partecipazione in qualità di relatore a convegni e congressi, conseguimento di premi e riconoscimenti per attività di ricerca, tirocini e stage formativi, ecc.) debitamente attestati (other qualifications related to previous activities (e.g. <i>scholarships, participation as a speaker at conferences and congresses, receipt of prizes and awards for research activity, internships and training periods, etc.</i>) duly certified)	studio, partecipazione in qualità di relatore a convegni e congressi, conseguimento di premi e riconoscimenti per attività di ricerca, tirocini e stage formativi, ecc.) debitamente attestati (other qualifications related to previous activities (e.g. <i>scholarships, participation as a speaker at conferences and congresses, receipt of prizes and awards for research activity, internships and training periods, etc.</i>) duly certified)
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Criteri di valutazione per il colloquio, che terrà conto anche degli aspetti motivazionali e attitudinali, nonché della capacità di esposizione e di sintesi

(Criteria for the interview, which will take also into account the candidate's motivation and aptitude, as well as their ability to present and summarize information):

- maturità scientifica;
- scientific maturity;
- preparazione dei candidati;
- preparation of the applicants.